Computers and Mathematics with Applications 62 (2011) 2259–2272

Contents lists available at SciVerse ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Pseudo almost automorphic solutions of some nonlinear integro-differential equations

Syed Abbas

School of Basic Sciences, Indian Institute of Technology Mandi, Mandi, H.P., 175001, India

ARTICLE INFO

Article history:
Received 30 March 2011
Received in revised form 6 July 2011
Accepted 7 July 2011

Keywords:
Pseudo almost automorphic function
Almost periodic function
Fractional powers of operators
Mild solution

ABSTRACT

In this paper we discuss the existence and uniqueness of a pseudo almost automorphic solution of an integro-differential equation in a Banach space X. We achieve our results using the methods of fractional powers of operators and the Banach fixed point theorem. These results are new and complement the existing ones.

1. Introduction

The concept of pseudo almost automorphic functions are a natural generalization of almost automorphic functions. The concept of the almost automorphic function was introduced by Bochner [1]. For more detail one can see the book by N'Guérékata [2] in which the author gave a very good overview of the theory of almost automorphic functions and their applications to differential equations. Almost automorphic solutions of various differential equations have been studied by many authors [2–6] and references therein. The concept of pseudo almost automorphy was suggested by N'Guérékata (see [2, page 40]) and developed by Xiao et al. [7]. The existence and uniqueness of pseudo almost automorphic solutions of differential equations have attracted the attention of many mathematicians in recent years [8–13,3,14].

Recently Xiao et al. [3] have shown the existence of an pseudo almost automorphic mild solution of the following differential equations

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = A(t)x(t) + \bar{f}(t, x(t)), \quad t \in \mathbb{R}, \tag{1.1}$$

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = A(t)x(t) + \bar{f}(t, x(t-h)), \quad t \in \mathbb{R}, \tag{1.2}$$

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = A(t)x(t) + \bar{f}(t, x(t), x[\alpha(t, x(t))]), \quad t \in \mathbb{R}, \tag{1.3}$$

in a Banach space. The cases $A(t) = A$ and $A(t+p) = A(t)$ for some positive p have been studied by many authors (see for instance [1,7] and references therein).

Motivated by the works mentioned above, we will study in this paper the problem of existence and uniqueness of pseudo almost automorphic solutions of the following integro-differential equation in a complex Banach space X,

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} + Au(t) = f(t, u(t), Ku(t)), \quad t \in \mathbb{R}, \ u \in PAA(X),$$

E-mail addresses: sabbas.iitk@gmail.com, abbas@iitmandi.ac.in.

0898-1221/$ – see front matter © 2011 Elsevier Ltd. All rights reserved.
doi:10.1016/j.camwa.2011.07.013

$$Ku(t) = \int_{-\infty}^{t} k(t-s)g(s, u(s))\mathrm{d}s, \tag{1.4}$$

where $f : \mathbb{R} \times X \times X \to X$, $g : \mathbb{R} \times X \to X$ and k satisfy $|k(t)| \leq C_k e^{-bt}$ for $t \geq 0$ and C_k, b are positive constants. We denote $PAA(X)$ the set of all pseudo almost automorphic functions from $\mathbb{R}$ to X. Further, we assume that $-A$ is the infinitesimal generator of an analytic semigroup $\{T(t), t \geq 0\}$ and the function $f(\cdot, u(\cdot), Ku(\cdot)) \in BC(\mathbb{R} \times \mathbb{X} \times \mathbb{X}, X)$, where BC denotes the set of all bounded continuous functions.

In [15] Diagana et al. have shown the existence of pseudo almost periodic solution using fractional powers of operators of the following differential equation

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} + Au(t) = \bar{g}(t, u(t)), \quad t \in \mathbb{R}, \tag{1.5}$$

in a Banach space X, where $\bar{g} : \mathbb{R} \times X \to X$, is a jointly continuous function, $-A$, the generator of an analytic semigroup. Also Bahaj and Sidki [16] studied the existence of almost periodic solution of differential equation (1.5) using fractional powers of operators.

Because the concept of pseudo almost automorphic functions is pretty new, there is not much literature available on the pseudo almost automorphic solution of functional, delay and patrial differential equations. Many authors have shown the existence of a pseudo almost automorphic mild solution under the Lipschitz condition on the forcing term. Here we show the existence and uniqueness of a pseudo almost automorphic solution of (1.4) using the method of fractional powers of linear operators and the Banach fixed point principle. At the end we give an example to illustrate the abstract results.

2. Preliminaries

We denote by $BC(\mathbb{R}, X)$ the space of all bounded continuous functions from $\mathbb{R}$ to X. It is a Banach space with the supremum norm

$$\|u\|_\infty = \sup_{t \in \mathbb{R}} \|u(t)\|.$$

Consider $B(X, Y)$ is the set of all bounded linear operators from X to Y. This is also a Banach space with norm

$$\|A\|_{B(X,Y)} = \sup_{x \in X, x \neq 0} \frac{\|Ax\|_Y}{\|x\|_X}.$$

Similarly $BC(\mathbb{R} \times X \times X \to X)$ is the Banach space of bounded continuous functions with supremum norm. Now we give some necessary definitions.

Fractional powers of operators:

It is possible to define fractional powers of A if $-A$ is the infinitesimal generator of an analytic semigroup $T(t)$ in a Banach space and $0 \in \rho(A)$. We define the fractional power $A^{-\alpha}$, for $\alpha > 0$ by

$$A^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} T(t)\mathrm{d}t.$$

It is well known that for $0 < \alpha \leq 1$, $A^\alpha : D(A^\alpha) \subset X \to X$ is a densely defined closed linear operator. $D(A^\alpha) \supset D(A)$ is the domain of A^α which is dense in X. For $f \in D(A^\alpha)$ we define the norm by

$$\|f\|_{D(A)} = \|f\| + \|A^\alpha f\|.$$

This graph norm is equivalent to the $\alpha - norm$ defined by $\|f\|_\alpha = \|A^\alpha f\|$. We denote X_α as the Banach space $D(A^\alpha)$ equipped with $\|\cdot\|_\alpha$. We observe that for $\alpha, \beta \geq 0$

$$A^{-\alpha-\beta} = A^{-\alpha}A^{-\beta}$$

and there exists a constant C such that $\|A^{-\alpha}\| \leq C$ for $0 < \alpha \leq 1$. The A^α is defined as the inverse of $A^{-\alpha}$. For a more detailed analysis on fractional powers of operators, the interested reader may consult [17].

Lemma 2.1. *Let $-A$ be the infinitesimal generator of an analytic semigroup $T(t)$. Then for $\alpha > 0$ and $0 \in \rho(A)$ we have,*

(i) *$T(t)A^\alpha x = A^\alpha T(t)x$ for every $x \in D(A^\alpha)$;*
(ii) *$T(t) : X \to D(A^\alpha)$ for every $t > 0$ and $\alpha \geq 0$;*
(iii) *for every $t > 0$ the operator $A^\alpha T(t)$ is bounded and*

$$\|A^\alpha T(t)\| \leq M_\alpha t^\alpha e^{-\delta t};$$

(iv) *for $0 < \alpha \leq 1$ and $x \in D(A^\alpha)$, we have*

$$\|(T(t)x - x)\| \leq C_\alpha t^\alpha \|A^\alpha x\|.$$

Definition 2.2. A continuous function $f : \mathbb{R} \to X$ is called almost automorphic if for every real sequence (s_n), there exists a subsequence (s_{n_k}) and a function $g \in L_{\text{loc}}(\mathbb{R}, X)$ such that

$$g(t) = \lim_{n\to\infty} f(t + s_{n_k})$$

is well defined for each $t \in \mathbb{R}$ and

$$\lim_{n\to\infty} g(t - s_{n_k}) = f(t)$$

for each $t \in \mathbb{R}$. Denote by $AA(X)$ the set of all such functions.

Definition 2.3. A continuous function $f : \mathbb{R} \times X \to X$ is called almost automorphic in t uniformly for x in compact subsets of X if for every compact subset K of X and every real sequence (s_n), there exists a subsequence $(s_{n_k})g \in L_{\text{loc}}(\mathbb{R} \times X, X)$ such that

$$g(t, x) = \lim_{n\to\infty} f(t + s_{n_k}, x)$$

is well defined for each $t \in \mathbb{R}$, $x \in K$ and

$$\lim_{n\to\infty} g(t - s_{n_k}, x) = f(t, x)$$

for each $t \in \mathbb{R}$, $x \in K$. Denote by $AA(\mathbb{R} \times X)$ the set of all such functions.

We denote by

$$AA_0(X) = \left\{ f \in BC(\mathbb{R}, X) : \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \|f(\xi)\| \mathrm{d}\xi = 0 \right\},$$

and by $AA_0(\mathbb{R} \times X \times X, X)$ the set of all continuous functions $f : \mathbb{R} \times X \times X \to X$ such that $f(., u, \chi) \in AA_0(X)$ and

$$\lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \|f(\xi, u, \chi)\| \mathrm{d}\xi = 0,$$

uniformly in $(u, \chi) \in X \times X$.

Definition 2.4. A function $f \in BC(\mathbb{R}, X)$ is called pseudo almost automorphic if it can be written as $f = f_1 + f_2$, where $f_1 \in AA(X)$ and $f_2 \in AA_0(X)$.

The functions f_1 and f_2 are called the almost automorphic and the ergodic perturbation components of f respectively. The set of all such functions will be denoted by $PAA(X)$.

Remark 2.5. A classical example of a pseudo almost automorphic function is

$$f(t) = \sin \frac{1}{2 + \cos t + \cos \sqrt{2}t} + \frac{1}{1 + t^2}, \quad t \in \mathbb{R}.$$

It is known that this function is not almost automorphic.

Definition 2.6. A continuous function $f : \mathbb{R} \times X \times X \to X$ is called pseudo almost automorphic in $t \in \mathbb{R}$ uniformly in $(x, \chi) \in X \times X$ if it can be written as $f = f_1 + f_2$, where $f_1 \in AA(\mathbb{R} \times X \times X, X)$ and $f_2 \in AA_0(\mathbb{R} \times X \times X, X)$.

We denote the set of all pseudo almost automorphic functions $f : \mathbb{R} \times X \times X \to X$ by $PAA(\mathbb{R} \times X \times X, X)$.

Definition 2.7. By pseudo almost automorphic mild solution $u : \mathbb{R} \to X$ we mean that $u \in PAA(X)$, and $u(t)$ satisfies

$$u(t) = T(t - a)u(a) + \int_a^t T(t - \xi)f(\xi, u(\xi), Ku(\xi))\mathrm{d}\xi, \tag{2.1}$$

for $t \geq a$.

It is easy to see that if $\|T(t)\| \leq M'\mathrm{e}^{-\delta t}$, then relation (2.1) can be replaced by

$$u(t) = \int_{-\infty}^{t} T(t - \xi)f(\xi, u(\xi), Ku(\xi))\mathrm{d}\xi,$$

(cf. for instance [2]).

Definition 2.8. By pseudo almost automorphic classical solution $u : \mathbb{R} \to X$ we mean that $u \in PAA(X)$, and $u(t) \in D(A)$ is continuously differentiable and satisfied (1.4).

Lemma 2.9. *Let $-A$ be the infinitesimal generator of an analytic semigroup $T(t)$. If $\hat{f}:\mathbb{R}\to X$ is a Holder continuous pseudo almost automorphic function, then the problem*

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t}+Au(t)=\hat{f}(t)\quad t\in\mathbb{R},$$

has a unique classical pseudo almost automorphic solution given by

$$u(t)=\int_{-\infty}^{t}T(t-\xi)\hat{f}(\xi)\mathrm{d}\xi.$$

Proof. As $-A$ generate an analytic semigroup, the solution of the Cauchy problem $\frac{\mathrm{d}u(t)}{\mathrm{d}t}+Au(t)$ with initial condition x_0 is given by $u(t)=T(t)x_0$ [17]. As $\hat{f}$ is Holder continuous, by using variation of parameter, we can represent the solution by

$$u(t)=T(t-a)x(a)+\int_{a}^{t}T(t-\xi)\hat{f}(\xi)\mathrm{d}\xi.$$

We assume that $\|T(t)\|\le M\mathrm{e}^{-\delta t}$, then we can write the solution as $u(t)=\int_{-\infty}^{t}T(t-\xi)\hat{f}(\xi)\mathrm{d}\xi$. Further our function $T(t-\xi)\hat{f}(\xi)$ is integrable over $(-\infty,t]$, thus using fundamental theorem of calculus, we conclude that $u(t)$ is continuous. But we already assumed that $\hat{f}$ is Holder continuous, we get that $u(t)$ is differentiable and hence it is a classical solution.

Our function $\hat{f}$ is pseudo almost automorphic, we can write this as $\hat{f}=\hat{f}_1+\hat{f}_2$, where $\hat{f}_1\in AA$ and $\hat{f}_2\in PAA_0$. As $\hat{f}_1$ is almost automorphic, for any sequence t_n, there exists a subsequence t_k such that

$$\hat{f}_1(t+t_k)\to\hat{f}^*(t),\qquad \hat{f}^*(t-t_k)\to\hat{f}_1(t).$$

Define

$$u_1(t)=\int_{-\infty}^{t}T(t-\xi)\hat{f}_1(\xi)\mathrm{d}\xi$$

and

$$u_2(t)=\int_{-\infty}^{t}T(t-\xi)\hat{f}_2(\xi)\mathrm{d}\xi.$$

Also denote

$$u^*(t)=\int_{-\infty}^{t}T(t-\xi)\hat{f}^*(\xi)\mathrm{d}\xi.$$

Thus

$$\begin{aligned}\|u_1(t+t_k)-u^*(t)\| &\le \int_{-\infty}^{t}\|T(t-\xi)\|\,\|\hat{f}_1(\xi+t_k)-\hat{f}^*(\xi)\|\mathrm{d}\xi\\ &\le \int_{-\infty}^{t}M\mathrm{e}^{-\delta(t-\xi)}\|\hat{f}_1(\xi+t_k)-\hat{f}^*(\xi)\|\mathrm{d}\xi\\ &\le \|\hat{f}_1(a+t_k)-\hat{f}^*(a)\|\int_{-\infty}^{t}M\mathrm{e}^{-\delta(t-\xi)}\mathrm{d}\xi\\ &\le \frac{M}{\delta}\|\hat{f}_1(a+t_k)-\hat{f}^*(a)\| \end{aligned}\tag{2.2}$$

for some $a\in(-\infty,t]$. Hence we get $u_1(t+t_k)\to u^*(t)$. Now consider

$$\begin{aligned}\|u^*(t-t_k)-u_1(t)\| &\le \int_{-\infty}^{t}\|T(t-\xi)\|\,\|\hat{f}^*(\xi-t_k)-\hat{f}_1(\xi)\|\mathrm{d}\xi\\ &\le \int_{-\infty}^{t}M\mathrm{e}^{-\delta(t-\xi)}\|\hat{f}^*(\xi-t_k)-\hat{f}_1(\xi)\|\mathrm{d}\xi\\ &\le \|\hat{f}^*(a-t_k)-\hat{f}_1(a)\|\int_{-\infty}^{t}M\mathrm{e}^{-\delta(t-\xi)}\mathrm{d}\xi\\ &\le \frac{M}{\delta}\|\hat{f}^*(b-t_k)-\hat{f}_1(b)\| \end{aligned}\tag{2.3}$$

for some $b\in(-\infty,t]$. Hence we get $u^*(t-t_k)\to u_1(t)$. From the above analysis, we conclude that u_1 is almost automorphic.

Next we consider

$$\int_{-r}^{r}\|u_2(t)\|\mathrm{d}t \le \int_{-r}^{r}\int_{-\infty}^{t} T(t-\xi)\hat{f}_2(\xi)\mathrm{d}\xi\mathrm{d}t \le J_1(r)+J_2(r),$$

where

$$J_1(r)=\int_{-r}^{r}\int_{-r}^{t}\|T(t-\xi)\|\,\|\hat{f}_2(\xi)\|\mathrm{d}\xi\mathrm{d}t, \qquad J_2(r)=\int_{-r}^{r}\int_{-\infty}^{-r}\|T(t-\xi)\|\,\|\hat{f}_2(\xi)\|\mathrm{d}\xi\mathrm{d}t.$$

By changing the order of integration in $J_1(r)$, we get

$$\begin{aligned}
J_1(r) &= \int_{-r}^{r}\int_{-r}^{t}\|T(t-\xi)\|\,\|\hat{f}_2(\xi)\|\mathrm{d}\xi\mathrm{d}t\\
&\le \int_{-r}^{r}\int_{\xi}^{r}\|T(t-\xi)\|\,\|\hat{f}_2(\xi)\|\mathrm{d}t\mathrm{d}\xi\\
&\le \int_{-r}^{r}\|\hat{f}_2(\xi)\|\left(\int_{\xi}^{r}\|T(t-\xi)\|\mathrm{d}t\right)\mathrm{d}\xi\\
&\le \int_{-r}^{r}\|\hat{f}_2(\xi)\|\left(\int_{\xi}^{r}M\mathrm{e}^{-\delta(t-\xi)}\mathrm{d}t\right)\mathrm{d}\xi\\
&\le \int_{-r}^{r}\|\hat{f}_2(\xi)\|\left(\int_{0}^{r-\xi}M\mathrm{e}^{-\delta t}\mathrm{d}t\right)\mathrm{d}\xi\\
&\le \int_{-r}^{r}\|\hat{f}_2(\xi)\|\left(\int_{0}^{\infty}M\mathrm{e}^{-\delta t}\mathrm{d}t\right)\mathrm{d}\xi\\
&\le \frac{M}{\delta}\int_{-r}^{r}\|\hat{f}_2(\xi)\|\mathrm{d}\xi.
\end{aligned} \tag{2.4}$$

Thus we have

$$\frac{1}{2r}J_1(r)\to 0 \quad \text{as } r\to\infty.$$

Now we consider $J_2(r)$,

$$\begin{aligned}
J_1(r) &= \int_{-r}^{r}\int_{-\infty}^{-r}\|T(t-\xi)\|\,\|\hat{f}_2(\xi)\|\mathrm{d}\xi\mathrm{d}t\\
&\le \int_{-r}^{r}\int_{t+r}^{\infty}\|T(s)\|\,\|\hat{f}_2(t-s)\|\mathrm{d}s\mathrm{d}t\\
&\le \int_{-r}^{r}\int_{2r}^{\infty}\|T(s)\|\,\|\hat{f}_2(t-s)\|\mathrm{d}s\mathrm{d}t\\
&\le 2r\|\hat{f}_2\|\int_{2r}^{\infty}\|T(s)\|\mathrm{d}s.
\end{aligned} \tag{2.5}$$

Hence as $r\to\infty$, we get $\frac{1}{2r}J_2(r)\to 0$. Combining both, we have

$$\frac{1}{2r}\int_{-r}^{r}\|u_2(t)\|\mathrm{d}t\to 0 \quad \text{as } r\to\infty.$$

From the above analysis we conclude that $u_1+u_2=u\in PAA$. □

Assumptions. We require some assumption for the existence of our result. We assume that,

(H1) The function $f:\mathbb{R}\times X\times X\to X$ is a pseudo almost automorphic in t uniformly in $(u,\chi)\in X\times X$, and there exists $L_f>0$, and $0\le a\le 1$ such that

$$\|f(t_1,u_1,\chi_1)-f(t_2,u_2,\chi_2)\|\le L_f[|t_1-t_2|^a+\|u_1-u_2\|_\alpha+\|\chi_1-\chi_2\|],$$

for each $(t_i,u_i,\chi_i)\in\mathbb{R}\times X\times X$, $i=1,2$;

(H2) $-A$ is the infinitesimal generator of an analytic semigroup $T(t)$ and $0\in\rho(A)$;

(H3) The function $g : \mathbb{R} \times X \to X$ is pseudo almost automorphic in t uniformly in $u \in X$, and satisfies

$$\|g(t, u_1) - g(t, u_2)\| \le L_g \|u_1 - u_2\|_\alpha$$

for each $u_1, u_2 \in X_\alpha$.

Throughout the paper we assume that these conditions hold.

Because $f,\ g$ are pseudo almost automorphic, assume that $f_1,\ f_2$ and $g_1,\ g_2$ are the automorphic and ergodic perturbations of f and g. From the definition of an almost automorphic function we have that for every sequence t_n there exists a subsequence t_{n_k} such that

$$\begin{aligned} &f_1(t + t_{n_k}, u, \chi) \to f_1^*(t, u, \chi), \\ &f_1^*(t - t_{n_k}, u, \chi) \to f_1(t, u, \chi), \quad (u, \chi) \in D, \end{aligned}$$

and

$$\begin{aligned} &g_1(t + t_{n_k}, u) \to g_1^*(t, u), \\ &g_1^*(t - t_{n_k}, u) \to g_1(t, u) \quad u \in E, \end{aligned}$$

where D, E are compact subsets of $X \times X$ and X respectively.

3. Pseudo almost automorphic solution

In this section we prove the existence and uniqueness result for pseudo almost automorphic solution of differential equation (1.4). Since $A^{-\alpha} \in \mathbf{B}(X, X)$, we have $(A^{-\alpha}\phi)(t)$ is pseudo almost automorphic.

Lemma 3.1. *If $g \in PAA(X)$, then $K(A^{-\alpha}u)(t) \in PAA(X)$.*

Proof. Because g is pseudo almost automorphic so it can be written as $g_1 + g_2$, where g_1 is almost automorphic and g_2 is the ergodic component.

$$\begin{aligned} KA^{-\alpha}u(t) &= \int_{-\infty}^{t} k(t-s)g(s, A^{-\alpha}u(s))\mathrm{d}s \\ &= \int_{-\infty}^{t} k(t-s)g_1(s, A^{-\alpha}u(s))\mathrm{d}s + \int_{-\infty}^{t} k(t-s)g_2(s, A^{-\alpha}u(s))\mathrm{d}s \\ &= K_1A^{-\alpha}u(t) + K_2A^{-\alpha}u(t). \end{aligned} \tag{3.1}$$

Define

$$K_1^*(A^{-\alpha}u)(t) = \int_{-\infty}^{t} k(t-s)g_1^*(s, A^{-\alpha}u(s))\mathrm{d}s. \tag{3.2}$$

For the subsequence t_{n_k} of t_n, we consider the following

$$\begin{aligned} K_1(A^{-\alpha}u)(t + t_{n_k}) - K_1^*(A^{-\alpha}u)(t) &= \int_{-\infty}^{t+t_{n_k}} k(t + t_{n_k} - s)g_1(s, A^{-\alpha}u(s))\mathrm{d}s - \int_{-\infty}^{t} k(t-s)g_1^*(s, A^{-\alpha}u(s))\mathrm{d}s \\ &= \int_{-\infty}^{t} k(t-s)g_1(s + t_{n_k}, A^{-\alpha}u(s + t_{n_k}))\mathrm{d}s \\ &\quad - \int_{-\infty}^{t} k(t-s)g_1^*(s, A^{-\alpha}u(s))\mathrm{d}s. \end{aligned} \tag{3.3}$$

Taking the norm of both sides of inequality (3.3) and using the mean value theorem for the integral, we have

$$\begin{aligned} \|K_1(A^{-\alpha}u)(t + t_{n_k}) - K_1^*(A^{-\alpha}u)(t)\| &\le \int_{-\infty}^{t} |k(t-s)| \|g_1(s + t_{n_k}, A^{-\alpha}u(s + t_{n_k})) - g_1^*(s, A^{-\alpha}u(s))\|\mathrm{d}s \\ &\le \|g_1(a + t_{n_k}, A^{-\alpha}u(a + t_{n_k})) - g_1^*(a, A^{-\alpha}u(a))\| \int_{-\infty}^{t} |k(t-s)|\mathrm{d}s \\ &\le \|g_1(a + t_{n_k}, A^{-\alpha}u(a + t_{n_k})) - g_1^*(a, A^{-\alpha}u(a))\| \int_{-\infty}^{t} |k(t-s)|\mathrm{d}s \\ &\le \frac{C_k}{b} \|g_1(a + t_{n_k}, A^{-\alpha}u(a + t_{n_k})) - g_1^*(a, A^{-\alpha}u(a))\|, \end{aligned} \tag{3.4}$$

for some $a \in (-\infty, t]$. Because g_1 is almost automorphic, so we get

$$\|K_1(A^{-\alpha}u)(t+t_{n_k}) - K_1^*(A^{-\alpha}u)(t)\| \to 0. \tag{3.5}$$

Now consider

$$\begin{aligned} K_1^*(A^{-\alpha}u)(t-t_{n_k}) - K_1(A^{-\alpha}u)(t) &= \int_{-\infty}^{t-t_{n_k}} k(t-t_{n_k}-s)g_1^*(s, A^{-\alpha}u(s))\mathrm{d}s - \int_{-\infty}^{t} k(t-s)g_1(s, A^{-\alpha}u(s))\mathrm{d}s \\ &= \int_{-\infty}^{t} k(t-s)g_1^*(s-t_{n_k}, A^{-\alpha}u(s-t_{n_k}))\mathrm{d}s - \int_{-\infty}^{t} k(t-s)g_1(s, A^{-\alpha}u(s))\mathrm{d}s. \end{aligned} \tag{3.6}$$

Taking the norm of both sides of inequality (3.6) and using the mean value theorem for the integral, we have

$$\begin{aligned} \|K_1^*(A^{-\alpha}u)(t-t_{n_k}) - K_1(A^{-\alpha}u)(t)\| &\le \int_{-\infty}^{t} |k(t-s)| \times \|g_1^*(s-t_{n_k}, A^{-\alpha}u(s-t_{n_k})) - g_1(s, A^{-\alpha}u(s))\|\mathrm{d}s \\ &\le \|g_1^*(b-t_{n_k}, A^{-\alpha}u(b-t_{n_k})) - g_1(b, A^{-\alpha}u(b))\| \int_{-\infty}^{t} |k(t-s)|\mathrm{d}s \\ &\le \frac{C_k}{b}\|g_1^*(b-t_{n_k}, A^{-\alpha}u(b-t_{n_k})) - g_1(b, A^{-\alpha}u(b))\|, \end{aligned} \tag{3.7}$$

for some $b \in (-\infty, t]$. Because g_1 is almost automorphic, so we get

$$\|K_1^*(A^{-\alpha}u)(t-t_{n_k}) - K_1(A^{-\alpha}u)(t)\| \to 0.$$

Hence $K_1(A^{-\alpha}u)(t)$ is almost automorphic. Next consider the following

$$\begin{aligned} \frac{1}{2r}\int_{-r}^{r} \|K_2(A^{-\alpha}u)(t)\|\mathrm{d}t &\le \frac{1}{2r}\int_{-r}^{r}\int_{-\infty}^{t} |k(t-s)|\|g_2(s, (A^{-\alpha}u)(s))\|\mathrm{d}s\mathrm{d}t \\ &\le I_1(r) + I_2(r), \end{aligned} \tag{3.8}$$

where

$$I_1(r) = \frac{1}{2r}\int_{-r}^{r}\int_{-r}^{t} |k(t-s)|\|g_2(s, (A^{-\alpha}u)(s))\|\mathrm{d}s\mathrm{d}t$$

and

$$I_2(r) = \frac{1}{2r}\int_{-r}^{r}\int_{-\infty}^{-r} |k(t-s)|\|g_2(s, (A^{-\alpha}u)(s))\|\mathrm{d}s\mathrm{d}t.$$

Considering I_1 and making a change of order, we get

$$\begin{aligned} I_1(r) &\le \frac{1}{2r}\int_{-r}^{r} \|g_2(s, (A^{-\alpha}u)(s))\| \left(\int_{s}^{r} |k(t-s)|\mathrm{d}t\right)\mathrm{d}s \\ &\le \frac{1}{2r}\int_{-r}^{r} \|g_2(s, (A^{-\alpha}u)(s))\| \left(\int_{0}^{r-s} |k(t)|\mathrm{d}t\right)\mathrm{d}s \\ &\le \frac{1}{2r}\int_{-r}^{r} \|g_2(s, (A^{-\alpha}u)(s))\| \left(\int_{0}^{\infty} |k(t)|\mathrm{d}t\right)\mathrm{d}s \\ &\le \frac{C_k}{2br}\int_{-r}^{r} \|g_2(s, (A^{-\alpha}u)(s))\|\mathrm{d}s. \end{aligned} \tag{3.9}$$

As $g_2 \in PAA_0$, we get

$$\lim_{r\to\infty} I_1(r) = 0.$$

For $I_2(r)$, we have

$$\begin{aligned} I_2(r) &\le \frac{1}{2r}\int_{-r}^{r}\int_{t+r}^{\infty} |k(s)|\|g_2(t-s, (A^{-\alpha}u)(t-s))\|\mathrm{d}s\mathrm{d}t \\ &\le \frac{1}{2r}\int_{-r}^{r}\int_{2r}^{\infty} |k(s)|\|g_2(t-s, (A^{-\alpha}u)(t-s))\|\mathrm{d}s\mathrm{d}t \\ &\le \|g_2\|_\infty \int_{2r}^{\infty} |k(s)|\mathrm{d}s. \end{aligned} \tag{3.10}$$

Thus

$$\lim_{r\to\infty} I_2(r) = 0.$$

Hence one can conclude that $K(A^{-\alpha}u)(t)$ is pseudo almost automorphic.

Define an operator F by

$$Fu(t) = \int_{-\infty}^{t} A^{\alpha}T(t-\xi)f(\xi, A^{-\alpha}u(\xi), K(A^{-\alpha}u(\xi)))\mathrm{d}\xi \tag{3.11}$$

from $PAA(X)$ to $PAA(X)$. □

Lemma 3.2. *The operator F is continuous.*

Proof. Consider the sequence $u_n \to u$. Now we need to show that $Fu_n \to Fu$. Taking the norm of the expression $Fu_n(t) - Fu(t)$ we have

$$\begin{aligned}\|Fu_n(t) - Fu(t)\| &\le \int_{-\infty}^{t} \|A^{\alpha}T(t-s)\|f(s, A^{-\alpha}u_n(s), KA^{-\alpha}u_n(s)) - \|f(s, A^{-\alpha}u(s), KA^{-\alpha}u(s))\|\mathrm{d}s \\ &\le \int_{-\infty}^{t} \|A^{\alpha}T(t-s)\|L_f(\|A^{-\alpha}u_n(s) - A^{-\alpha}u(s)\|_{\alpha} + \|KA^{-\alpha}u_n(s) - KA^{-\alpha}u(s)\|)\mathrm{d}s.\end{aligned}$$

Also note that

$$\begin{aligned}\|KA^{-\alpha}u_n(s) - KA^{-\alpha}u(s)\| &\le \int_{-\infty}^{t} |k(t-s)| \times \|g(s, A^{-\alpha}u_n(s)) - g(s, A^{-\alpha}u(s))\|\mathrm{d}s \\ &\le \int_{-\infty}^{t} |k(t-s)|L_g\|A^{-\alpha}u_n(s) - A^{-\alpha}u(s)\|_{BC(\mathbb{R},X_{\alpha})}\mathrm{d}s \\ &\le \frac{C_kL_g}{b}\|A^{-\alpha}u_n - A^{-\alpha}u\|_{\alpha}.\end{aligned}$$

Thus we have

$$\begin{aligned}\|Fu_n(t) - Fu(t)\| &\le L_f\left(1 + \frac{C_kL_g}{b}\right)\|A^{-\alpha}u_n - A^{-\alpha}u\|_{\alpha}\int_{-\infty}^{t}\|A^{\alpha}T(t-s)\|\mathrm{d}s \\ &\le L_f\left(1 + \frac{C_kL_g}{b}\right)\|A^{-\alpha}u_n - A^{-\alpha}u\|_{\alpha}\int_{0}^{\infty}\|A^{\alpha}T(s)\|\mathrm{d}s \\ &\le L_f\left(1 + \frac{C_kL_g}{b}\right)\|A^{-\alpha}u_n - A^{-\alpha}u\|_{\alpha}\int_{0}^{\infty}M_{\alpha}s^{-\alpha}\mathrm{e}^{-\delta s}\mathrm{d}s \\ &\le M_{\alpha}L_f\delta^{1-\alpha}\Gamma(1-\alpha)\left(1 + \frac{C_kL_g}{b}\right)\|A^{-\alpha}u_n - A^{-\alpha}u\|_{\alpha} \\ &\le C^*\|A^{-\alpha}u_n - A^{-\alpha}u\|_{\alpha},\end{aligned} \tag{3.12}$$

where $C^* = M_{\alpha}L_f\delta^{1-\alpha}\Gamma(1-\alpha)\left(1 + \frac{C_kL_g}{b}\right)$. Taking the supremum of both sides of the above inequality, we get

$$\|Fu_n - Fu\| \le C^*\|A^{-\alpha}u_n - A^{-\alpha}u\|_{\alpha} \le C^*\|u_n - u\|.$$

From the above relation it is easy to deduce that $Fu_n \to Fu$ and hence F is continuous. □

Lemma 3.3. *The operator F is bounded.*

Proof. We observe that the result of Liang et al. [18] of the composition theorem for a function $f \in PAA(\mathbb{R} \times X)$ may be extended for a function $f \in PAA(\mathbb{R} \times X \times X)$. For $u \in PAA(X)$, by the composition theorem for pseudo-almost automorphic function [18], $\phi(\cdot) = f(\cdot, A^{-\alpha}u(\cdot), K(A^{-\alpha}u(\cdot)))$is pseudo almost automorphic. Now taking the norm of the operator F, we have

$$\begin{aligned}\|(Fu)(t)\| &\le \int_{-\infty}^{t} M_{\alpha}(t-\xi)^{-\alpha}\mathrm{e}^{-\delta(t-\xi)}\|f(\xi, A^{-\alpha}u(\xi), K(A^{-\alpha}u(\xi)))\|\mathrm{d}\xi \\ &\le M_{\alpha}\int_{-\infty}^{t}(t-\xi)^{-\alpha}\mathrm{e}^{-\delta(t-\xi)}[\|f(\xi, A^{-\alpha}u(\xi), K(A^{-\alpha}u(\xi))) - f(\xi, A^{-\alpha}u(\xi), 0)\| + \|f(\xi, A^{-\alpha}u(\xi), 0)\|]\mathrm{d}\xi\end{aligned}$$

$$\leq M_\alpha N \int_{-\infty}^{t} (t-\xi)^{-\alpha} e^{-\delta(t-\xi)} d\xi$$
$$= M_\alpha N \int_{0}^{\infty} \eta^{-\alpha} e^{-\delta\eta} d\eta < \infty,$$

for some positive constant N, which is the bound of the function f. Hence Fu is bounded.

Define $\phi(\cdot) = f(\cdot, A^{-\alpha}u(\cdot), K(A^{-\alpha}u(\cdot)))$, as we mention above that if $u \in PAA(X)$ then $\phi \in PAA(X)$.

Hence $\phi = \phi_1 + \phi_2$ with $\phi_1 \in AA(X)$ and $\phi_2 \in AA_0(X)$.

Because f is pseudo almost automorphic we have

$$\begin{aligned} Fu(t) &= \int_{-\infty}^{t} A^\alpha T(t-\xi) f(\xi, A^{-\alpha}u(\xi), K(A^{-\alpha}u(\xi))) d\xi \\ &= \int_{-\infty}^{t} A^\alpha T(t-\xi) f_1(\xi, A^{-\alpha}u(\xi), K(A^{-\alpha}u(\xi))) d\xi + \int_{-\infty}^{t} A^\alpha T(t-\xi) f_2(\xi, A^{-\alpha}u(\xi), K(A^{-\alpha}u(\xi))) d\xi \\ &= F_1u(t) + F_2u(t). \end{aligned}$$

Next we present some results related to the pseudo almost automorphy of our operator F. The ϕ_1 and ϕ_2 defined above are $\phi_1(t) = f_1(t, A^{-\alpha}u(t), K(A^{-\alpha}u(t)))$ and $\phi_2(t) = f_2(t, A^{-\alpha}u(t), K(A^{-\alpha}u(t)))$ for fixed u. These notations will actually simplify the calculations. □

Lemma 3.4. *The map defined by,*

$$F_1\phi_1(t) = \int_{-\infty}^{t} A^\alpha T(t-\xi)\phi_1(\xi) d\xi \tag{3.13}$$

is in $AA(X)$ if ϕ_1 is in $AA(X)$.

Proof. Because ϕ_1 is almost automorphic, then for any sequence t_n one has a subsequence t_{n_k} such that

$$\phi_1(t + t_{n_k}) \to \phi_1^*(t), \qquad \phi_1^*(t - t_{n_k}) \to \phi_1(t).$$

Define

$$F_1^*\phi_1(t) = \int_{-\infty}^{t} A^\alpha T(t-\xi)\phi_1^*(\xi) d\xi.$$

Consider

$$\begin{aligned} F_1\phi_1(t + t_{n_k}) - F_1^*\phi_1(t) &= \int_{-\infty}^{t+t_{n_k}} A^\alpha T(t + t_{n_k} - \xi)\phi_1(\xi) d\xi - \int_{-\infty}^{t} A^\alpha T(t-\xi)\phi_1^*(\xi) d\xi \\ &= \int_{-\infty}^{t} A^\alpha T(t-\xi)[\phi_1(\xi + t_{n_k}) - \phi_1^*(\xi)] d\xi. \end{aligned} \tag{3.14}$$

Hence taking the norm of Eq. (3.14) and using the mean value theorem for integrals, for some $a \in (-\infty, t]$, one has

$$\begin{aligned} \|F_1\phi_1(t + t_{n_k}) - F_1^*\phi_1(t)\| &\leq \int_{-\infty}^{t} \|A^\alpha T(t-\xi)\| \, \|\phi_1(\xi + t_{n_k}) - \phi_1^*(\xi)\| d\xi \\ &\leq \|\phi_1(a + t_{n_k}) - \phi_1^*(a)\| \int_{-\infty}^{t} \|A^\alpha T(t-\xi)\| d\xi \\ &\leq \epsilon \int_{-\infty}^{t} \|A^\alpha T(t-\xi)\| d\xi \\ &\leq \epsilon M_\alpha \int_{0}^{\infty} \eta^{-\alpha} e^{-\delta\eta} d\eta \\ &\leq \epsilon'. \end{aligned}$$

Now consider

$$\begin{aligned} F_1^*\phi_1(t - t_{n_k}) - F_1\phi_1(t) &= \int_{-\infty}^{t-t_{n_k}} A^\alpha T(t - t_{n_k} - \xi)\phi_1^*(\xi) d\xi - \int_{-\infty}^{t} A^\alpha T(t-\xi)\phi_1(\xi) d\xi \\ &= \int_{-\infty}^{t} A^\alpha T(t-\xi)[\phi_1^*(\xi - t_{n_k}) - \phi_1(\xi)] d\xi. \end{aligned} \tag{3.15}$$

Hence taking the norm of Eq. (3.15) and using the mean value theorem for integrals, for some $b \in (-\infty, t]$, we have

$$\begin{aligned}\|F_1^*\phi_1(t-t_{n_k}) - F_1\phi_1(t)\| &\le \int_{-\infty}^{t} \|A^\alpha T(t-\xi)\|\,\|\phi_1^*(\xi-t_{n_k}) - \phi_1(\xi)\|\mathrm{d}\xi \\ &\le \|\phi_1^*(b-t_{n_k}) - \phi_1(b)\| \int_{-\infty}^{t} \|A^\alpha T(t-\xi)\|\mathrm{d}\xi \\ &\le \epsilon \int_{-\infty}^{t} \|A^\alpha T(t-\xi)\|\mathrm{d}\xi \\ &\le \epsilon \int_0^\infty \eta^{-\alpha} \mathrm{e}^{-\delta\eta}\mathrm{d}\eta \\ &\le \epsilon'.\end{aligned}$$

Hence $F_1(\phi_1)$ is almost automorphic. □

Lemma 3.5. *The map F_2 defined by,*

$$F_2\phi_2(t) = \int_{-\infty}^{t} A^{-\alpha} T(t-\xi)\phi_2(\xi)\mathrm{d}\xi \tag{3.16}$$

is in $AA_0(X)$ for ϕ_2 belongs to $AA_0(X)$.

Proof. It is easy to see that $F_2\phi_2(t)$ is bounded continuous.

$$\frac{1}{2r}\int_{-r}^{r} \|F_2\phi_2(t)\|\mathrm{d}t \le \frac{1}{2r}\int_{-r}^{r}\int_{-\infty}^{t} \|A^\alpha T(t-\xi)\|\,\|\phi_2(\xi)\|\mathrm{d}\xi\mathrm{d}t \le I_3(r) + I_4(r)$$

where

$$I_3(r) = \frac{1}{2r}\int_{-r}^{r}\int_{-r}^{t} \|A^\alpha T(t-\xi)\|\,\|\phi_2(\xi)\|\mathrm{d}\xi\mathrm{d}t$$

and

$$I_4(r) = \frac{1}{2r}\int_{-r}^{r}\int_{-\infty}^{-r} \|A^\alpha T(t-\xi)\|\,\|\phi_2(\xi)\|\mathrm{d}\xi\mathrm{d}t.$$

By changing the order of integration in I_3, we have

$$\begin{aligned} I_3(r) &\le \frac{1}{2r}\int_{-r}^{r} \|\phi_2(\xi)\| \left(\int_{\xi}^{r} \|A^\alpha T(t-\xi)\|\mathrm{d}t\right)\mathrm{d}\xi \\ &\le \frac{1}{2r}\int_{-r}^{r} \|\phi_2(\xi)\| \left(\int_{0}^{r-s} \|A^\alpha T(t)\|\mathrm{d}t\right)\mathrm{d}\xi \\ &\le \frac{1}{2r}\int_{-r}^{r} \|\phi_2(\xi)\| \left(\int_{0}^{\infty} \|A^\alpha T(t)\|\mathrm{d}t\right)\mathrm{d}\xi \\ &\le \frac{1}{2r}\int_{-r}^{r} \|\phi_2(\xi)\| \left(\int_{0}^{\infty} M_\alpha t^{-\alpha}\mathrm{e}^{-\delta t}\mathrm{d}t\right)\mathrm{d}\xi \\ &\le M_\alpha \delta^{1-\alpha}\Gamma(1-\alpha)\frac{1}{2r}\int_{-r}^{r} \|\phi_2(\xi)\|\mathrm{d}\xi. \end{aligned} \tag{3.17}$$

The above calculations imply that

$$\lim_{r\to\infty} I_3(r) = 0.$$

Now consider

$$\begin{aligned} I_4(r) &\le \frac{1}{2r}\int_{-r}^{r}\int_{t+r}^{\infty} \|A^\alpha T(s)\|\,\|\phi_2(t-s)\|\mathrm{d}s\mathrm{d}t \\ &\le \frac{1}{2r}\int_{-r}^{r}\int_{2r}^{\infty} \|A^\alpha T(s)\|\,\|\phi_2(t-s)\|\mathrm{d}s\mathrm{d}t \\ &\le \|\phi_2\|_\infty \int_{2r}^{\infty} \|A^\alpha T(s)\|\mathrm{d}s. \end{aligned} \tag{3.18}$$

From the above analysis we get

$$\lim_{r\to\infty} I_4(r) = 0.$$

So we have

$$\lim_{r\to\infty} \frac{1}{2r}\int_{-r}^{r} \|F_2(\phi_2)t\|\mathrm{d}t = 0.$$

We assumed in the beginning that $|k(t)| \le C_k \mathrm{e}^{-bt}$, hence we have $\int_0^\infty |k(s)|\mathrm{d}s \le \frac{C_k}{b}$. Denote $BC(\mathbb{R}, \mathbb{X})$, the space of all bounded, continuous functions from $\mathbb{R}$ to X. □

Theorem 3.6. *Assume that $f \in PAA(\mathbb{R} \times X \times BC(\mathbb{R}, X), X)$ is Lipschitz continuous and $-A$ is the infinitesimal generator of an analytic semigroup $T(t)$. Then Eq. (1.4) has a unique pseudo almost automorphic solution if $\Lambda = CM_\alpha \left(L_f + \frac{C_k}{b} L_g\right)\delta^{1-\alpha}\Gamma(1-\alpha) < 1$.*

Proof. From Lemmas 3.3–3.5 it follows that the operator F is well defined, that is it maps $PAA(X)$ to $PAA(X)$. For $u, v \in X$ consider,

$$\begin{aligned}\|(Fu)(t) - (Fv)(t)\| &= \left\|\int_{-\infty}^{t} A^\alpha T(t-\xi)(f(\xi, A^{-\alpha}u(\xi), K(A^{-\alpha}u(\xi)))) - f(\xi, A^{-\alpha}v(\xi), K(A^{-\alpha}v(\xi)))\mathrm{d}\xi\right\| \\ &\le \int_{-\infty}^{t} M_\alpha (t-\xi)^{-\alpha}\mathrm{e}^{-\delta(t-\xi)} L_f[\|A^{-\alpha}u(\xi) - A^{-\alpha}v(\xi)\|_\alpha \\ &\quad + \|K(A^{-\alpha}u(\xi)) - K(A^{-\alpha}v(\xi))\|]\mathrm{d}\xi. \end{aligned} \tag{3.19}$$

Consider

$$\begin{aligned}\|K(A^{-\alpha}u(t)) - K(A^{-\alpha}v(t))\| &\le \int_{-\infty}^{t} |k(t-s)|\|g(s, A^{-\alpha}u(s)) - g(s, A^{-\alpha}v(s))\|\mathrm{d}s \\ &\le \int_{-\infty}^{t} |k(t-s)| L_g \|A^{-\alpha}u(s) - A^{-\alpha}v(s)\|_\alpha \mathrm{d}s \\ &\le L_g \left(\int_{-\infty}^{t} |k(t-s)|\mathrm{d}s\right)\|u-v\|_\infty \\ &\le L_g\|u-v\|_\infty \int_{-\infty}^{t} |k(t-s)|\mathrm{d}s \\ &\le L_g\|u-v\|_\infty \int_{0}^{\infty} |k(s)|\mathrm{d}s \quad (\text{Using the transformation } t-s=s_1) \\ &\le \frac{C_k}{b} L_g \|u-v\|_\infty. \end{aligned}$$

Using the above estimate, inequality (3.19) becomes

$$\begin{aligned}\|(Fu)(t) - (Fv)(t)\| &< M_\alpha\left(L_f + \frac{C_k}{b}L_g\right)\|u-v\|_\infty \int_{-\infty}^{t} (t-\xi)^{-\alpha}\mathrm{e}^{-\delta(t-\xi)}\mathrm{d}\xi \\ &\le M_\alpha\left(L_f + \frac{C_k}{b}L_g\right)\|u-v\|_\infty \delta^{1-\alpha}\Gamma(1-\alpha). \end{aligned}$$

So for $\Lambda < 1$, by the Banach fixed-point principle the operator F has a unique fixed point ϕ_0 such that

$$\phi_0(t) = \int_{-\infty}^{t} A^\alpha T(t-\xi)(f(\xi, A^{-\alpha}\phi_0(\xi), K(A^{-\alpha}\phi_0(\xi))))\mathrm{d}\xi.$$

Since A^α is closed [17],

$$\phi_0(t) = A^\alpha \int_{-\infty}^{t} T(t-\xi)(f(\xi, A^{-\alpha}\phi_0(\xi), K(A^{-\alpha}\phi_0(\xi))))\mathrm{d}\xi.$$

Applying $A^{-\alpha}$ to both sides we get,

$$A^{-\alpha}\phi_0(t) = \int_{-\infty}^{t} T(t-\xi)(f(\xi, A^{-\alpha}\phi_0(\xi), K(A^{-\alpha}\phi_0(\xi))))\mathrm{d}\xi.$$

Our next task is to show that $\phi_0(t)$ is continuous.
Consider

$$\|\phi_0(t+h)-\phi_0(t)\| \le \left\| \int_{-\infty}^{t+h} A^\alpha T(t+h-\xi) f(\xi, A^{-\alpha}\phi_0(\xi), K(A^{-\alpha}\phi_0(\xi)))\mathrm{d}\xi - \int_{-\infty}^{t} A^\alpha T(t-\xi) f(\xi, A^{-\alpha}\phi_0(\xi), K(A^{-\alpha}\phi_0(\xi)))\mathrm{d}\xi \right\|$$
$$\le \left\| \int_{-\infty}^{t} A^\alpha T(t-\xi) \left(f(\xi+h, A^{-\alpha}\phi_0(\xi+h), K(A^{-\alpha}\phi_0(\xi+h))) - f(\xi, A^{-\alpha}\phi_0(\xi), K(A^{-\alpha}\phi_0(\xi))) \right)\mathrm{d}\xi \right\|. \tag{3.20}$$

Using the assumption on f and bound on K, we have

$$\begin{aligned}
&\|\phi_0(t+h)-\phi_0(t)\| \\
&\le \int_{-\infty}^{t} \|A^\alpha T(t-\xi)\| \, \|f(\xi+h, A^{-\alpha}\phi_0(\xi+h), K(A^{-\alpha}\phi_0(\xi+h))) - f(\xi, A^{-\alpha}\phi_0(\xi), K(A^{-\alpha}\phi_0(\xi)))\|\mathrm{d}\xi \\
&\le L_f \int_{-\infty}^{t} \|A^\alpha T(t-\xi)\| (|h|^a + \|A^{-\alpha}\phi_0(\xi+h) - A^{-\alpha}\phi_0(\xi)\|_\alpha + \|K(A^{-\alpha}\phi_0(\xi+h)) - K(A^{-\alpha}\phi_0(\xi))\|)\mathrm{d}\xi \\
&\le L_f \int_{-\infty}^{t} \|A^\alpha T(t-\xi)\| (|h|^a + \|\phi_0(\xi+h) - \phi_0(\xi)\| + \|K(A^{-\alpha}\phi_0(\xi+h)) - K(A^{-\alpha}\phi_0(\xi))\|)\mathrm{d}\xi \\
&\le L_f \int_{-\infty}^{t} \|A^\alpha T(t-\xi)\| \left(|h|^a + \|\phi_0(\xi+h) - \phi_0(\xi)\| + \frac{C_k L_g}{b} \|A^{-\alpha}\phi_0(\xi+h) - A^{-\alpha}\phi_0(\xi)\|_\alpha \right)\mathrm{d}\xi \\
&\le L_f \int_{-\infty}^{t} \|A^\alpha T(t-\xi)\| \left(|h|^a + \|\phi_0(\xi+h) - \phi_0(\xi)\| + \frac{C_k L_g}{b} \|\phi_0(\xi+h) - \phi_0(\xi)\| \right)\mathrm{d}\xi.
\end{aligned} \tag{3.21}$$

Denote $\Phi(t) = \|\phi_0(t+h)-\phi_0(t)\|$, we get

$$\begin{aligned}
\Phi(t) &\le L_f |h|^a \int_{-\infty}^{t} \|A^\alpha T(t-\xi)\|\mathrm{d}\xi + L_f \left(1+\frac{C_k L_g}{b}\right) \int_{-\infty}^{t} \|A^\alpha T(t-\xi)\| \Phi(\xi)\mathrm{d}\xi \\
&\le L_f |h|^a M_\alpha \delta^{1-\alpha}\Gamma(1-\alpha) + L_f \left(1+\frac{C_k L_g}{b}\right) \int_{-\infty}^{t} \|A^\alpha T(t-\xi)\| \Phi(\xi)\mathrm{d}\xi.
\end{aligned}$$

Using Gronwall's inequality, we have

$$\Phi(t) \le L_f |h|^a M_\alpha \delta^{1-\alpha}\Gamma(1-\alpha) \mathrm{e}^{L_f\left(1+\frac{C_k L_g}{b}\right)\int_{-\infty}^{t} \|A^\alpha T(t-\xi)\|\mathrm{d}\xi}.$$

After further evaluation, we get

$$\Phi(t) \le L_f |h|^a M_\alpha \delta^{1-\alpha}\Gamma(1-\alpha) \mathrm{e}^{L_f\left(1+\frac{C_k L_g}{b}\right) M_\alpha \delta^{1-\alpha}\Gamma(1-\alpha)}.$$

Hence for $h \to 0$, we get $\Phi(t) \to 0$, which is the same as

$$\|\phi_0(t+h)-\phi_0(t)\| \to 0.$$

Thus $\phi_0(t)$ is continuous.

By the assumption on f, we have

$$\begin{aligned}
&\|f(t, A^{-\alpha}\phi_0(t), K(A^{-\alpha}\phi_0(t))) - f(s, A^{-\alpha}\phi_0(s), K(A^{-\alpha}\phi_0(s)))\| \\
&\quad \le L_f(|t-s|^a + \|\phi_0(t)-\phi_0(s)\| + \|K(A^{-\alpha}\phi_0(t)) - K(A^{-\alpha}\phi_0(s))\|_\alpha).
\end{aligned}$$

Hence $f(t, A^{-\alpha}\phi_0(t), K(A^{-\alpha}\phi_0(t)))$ is Holder continuous on $\mathbb{R}$. Consider the equation

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} + Au(t) = f(t, A^{-\alpha}\phi_0(t), K(A^{-\alpha}\phi_0)(t)). \tag{3.22}$$

Using Lemma 2.9 it follows that (3.22) has a unique solution given by

$$u(t)=\int_{-\infty}^{t}T(t-\xi)f(\xi,A^{-\alpha}\phi_0(\xi),K(A^{-\alpha}\phi_0(\xi)))\mathrm{d}\xi.$$

Furthermore we have $u(t)\in D(A)$ for all $t\in\mathbb{R}$. Applying A^{α} to both sides we get

$$A^{\alpha}u(t)=\int_{-\infty}^{t}A^{\alpha}T(t-\xi)f(\xi,A^{-\alpha}\phi_0(\xi),K(A^{-\alpha}\phi_0(\xi)))\mathrm{d}\xi.$$

It is clear that $u(t)=A^{-\alpha}\phi_0(t)$ is the solution of (1.4), which is a pseudo almost automorphic solution. $\square$

Remark. By the similar method one can also show the existence of a pseudo almost automorphic solution of the following neutral functional differential equation in a complex Banach space X,

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t}=Au(t)+\frac{\mathrm{d}}{\mathrm{d}t}\bar{F_1}(t,u(t-\tau(t)))+\bar{F_2}(t,u(t),u(t-\tau(t))),\quad t\in\mathbb{R},\ u\in PAA(X) \tag{3.23}$$

where F_1, F_2 are pseudo almost automorphic functions. In this case the operator F is defined by

$$(Fu)(t)=\bar{F_1}(t,u(t-\tau(t)))+\int_{-\infty}^{t}T(t-s)\bar{F_2}(s,u(s),u(s-\tau(s)))\mathrm{d}s.$$

4. Example

Consider the partial differential equation,

$$\frac{\partial w(t,x)}{\partial t}-\frac{\partial^2 w(t,x)}{\partial x^2}=f(t,x,w(t,x),Kw(t,x)),\quad t\in\mathbb{R},\ x\in(0,1)$$

$$Kw(t,x)=\int_{-\infty}^{t}k(t-s)g(s,x,w(s,x))\mathrm{d}s, \tag{4.1}$$

$$w(t,0)=w(t,1)=0, \tag{4.2}$$

where k is a real valued function satisfying $|k(t)|\le C_k\mathrm{e}^{-bt}$ for $t\ge 0$ and C_k, b are positive constants. The map f is defined from $\mathbb{R}\times(0,1)\times\mathbb{R}\times\mathbb{R}$ into $\mathbb{R}$ and g is defined from $\mathbb{R}\times(0,1)\times\mathbb{R}$ into $\mathbb{R}$. For $u\in D(A)=\{u\in H_0^1(0,1)\cap H^2(0,1):u''\in H\}$, we define an operator A as follows,

$$Au=-u''.$$

The operator A is the infinitesimal generator of an analytic semigroup $\{T(t):t\ge 0\}$, and also self adjoint [17]. Now take $\alpha=\frac{1}{2}$, so $D(A^{1/2})$ is a Banach space endowed with the norm,

$$\|x\|_{1/2}=\|A^{1/2}x\|,\quad x\in D(A^{1/2}),$$

we call this space $X_{1/2}$. Denote

$$\mathcal{C}^{1/2}=C(\mathbb{R},D(A^{1/2})),$$

endowed with the sup norm

$$\|\psi\|_{\mathcal{C}^{1/2}}=\sup_{t\in\mathbb{R}}\|\psi(\eta)\|_{1/2},\quad \psi\in\mathcal{C}^{1/2}.$$

For $u\in D(A)$ and $\lambda\in\mathbb{R}$, with $Au=-u''=\lambda u$, we have

$$\|u'\|_{L^2}^2=(-u'',u)=(Au,u)=(\lambda u,u)=\lambda\|u\|_{L^2}^2,$$

so $\lambda>0$. A solution u of $Au=\lambda u$ is of the form

$$u(x)=C\cos(\sqrt{\lambda}x)+D\sin(\sqrt{\lambda}x)$$

using conditions $u(0)=u(1)=0$ we get $C=0$ and $\lambda=\lambda_n=n^2\pi^2$, $n\in\mathbf{N}$. Thus we get the corresponding solution for each $n\in\mathbf{N}$

$$u_n(x)=D\sin(\sqrt{\lambda_n}x).$$

We may easily observe that $(u_n,u_m)=1$ for $n=m$ and $(u_n,u_m)=0$ for $n\ne m$. So we get $D=\sqrt{2}$. For $u\in D(A)$ there exists a sequence of real numbers $\{\alpha_n\}$ such that

$$u(x)=\sum_{n\in\mathbf{N}}\alpha_n u_n(x),\qquad \sum_{n\in\mathbf{N}}(\alpha_n)^2<\infty,\qquad \sum_{n\in\mathbf{N}}(\lambda_n)^2(\alpha_n)^2<\infty.$$

We get

$$A^{1/2}u(x) = \sum_{n\in\mathbf{N}} \sqrt{\lambda_n}\alpha_n u_n(x), \quad u \in D(A^{1/2});$$

that is $\sum_{n\in\mathbf{N}} \lambda_n(\alpha_n)^2 < \infty$. Consider $u(t)(x) = w(t,x),\ t \in \mathbb{R},\ x \in (0,1)$ and $f(t,u(t),Ku(t))(x) = f(t,x,u(t)x,Ku(t)x),\ t\in\mathbb{R},\ x\in(0,1)$. Then the Eq. (4.2) may be written as the following abstract form in $X = L^2((0,1),\mathbb{R})$,

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} + Au(t) = f(t,u(t),Ku(t)), \quad t\in\mathbb{R},\ u\in PAA(X),$$
$$Ku(t) = \int_{-\infty}^{t} k(t-s)g(s,u(s))\mathrm{d}s.$$

As an example for K we may take

$$g(t,u) = u\sin\frac{1}{2+\cos t+\cos\sqrt{2}t} + \max_{m\in Z}\{\mathrm{e}^{-(t+m^2)^2}\}\sin u, \quad t\in\mathbb{R}$$

as pseudo almost automorphic. Calculating the norm of g we have

$$\begin{aligned}\|g(t,u)-g(t,v)\| &\le \left\| u\sin\frac{1}{2+\cos t+\cos\sqrt{2}t} - v\sin\frac{1}{2+\cos t+\cos\sqrt{2}t}\right\| + \|\sin u - \sin v\| \\ &\le \|u-v\| + \|u-v\| \le 2\|u-v\|. \end{aligned} \tag{4.3}$$

Thus in this case our constant $L_g = 2$. Also we have $\alpha = \frac{1}{2}$, hence the constant $\Lambda = M_\alpha(L_f + ML_g)\delta^{1-\alpha}\Gamma(1-\alpha)$ becomes $M_\alpha(L_f+2M)\delta^{1/2}\sqrt{\pi}$. Here we assume that f satisfy all the assumptions of Theorem 3.6 with Lipschitz constant L_f. Now the condition for the existence of a pseudo almost automorphic solution of Eq. (4.2) in the reference of Theorem 3.6 is given by $M_\alpha(L_f+2M)\sqrt{\delta\pi} < 1$.

Acknowledgment

The author would like to thank the anonymous referee for his/her comments and suggestions which helped me improve the manuscript.

References

[1] S. Bochner, Continuous mappings of almost automorphic and almost automorphic functions, Proc. Nat. Acad. Sci. USA 52 (1964) 907–910.
[2] G.M. N'Guérékata, Topics in Almost Automorphy, Springer-Verlag, New York, 2005.
[3] T.J. Xiao, X.X. Zhu, J. Liang, Pseudo almost automorphic mild solutions to nonautonomous differential equations and applications, Nonlinear Anal. TMA 70 (11) (2009) 4079–4085.
[4] S. Zaidman, Abstract Differential Equations, Pitman Publishing, San Francisco, London, Melbourne, 1979.
[5] S. Zaidman, Topics in Abstract Differential Equations—1, in: Pitman Research Notes in Mathematics Series, vol. 304, Longman Sci. Tech., UK, 1994.
[6] S. Zaidman, Topics in Abstract Differential Equations—2, in: Pitman Research Notes in Mathematics, vol. 321, 1995.
[7] T.J. Xiao, J. Liang, J. Zhang, Pseudo almost automorphic solutions to semilinear differential equations in Banach space, Semigroup Forum 76 (2008) 518–524.
[8] S. Abbas, Pseudo almost automorphic solutions of fractional order neutral differential equation, Semigroup Forum 81 (3) (2010) 393–404.
[9] T. Diagana, Pseudo-almost automorphic solutions to some classes of nonautonomous partial evolution equations, Difference Equ. Appl. 1 (4) (2009) 561–582.
[10] H.S. Ding, W. Long, G.M. N'Guérékata, Almost automorphic solutions of nonautonomous evolution equations, Nonlinear Anal. TMA 70 (12) (2009) 4158–4164.
[11] K. Ezzinbi, S. Fatajou, G.M. N'Guérékata, Pseudo almost automorphic solutions for some partial functional differential equations with infinite delay, Appl. Anal. 87 (5) (2008) 591–605.
[12] Khalil. Ezzinbi, Samir. Fatajou, G. N'Guérékata, Pseudo almost-automorphic solutions to some neutral partial functional differential equations in Banach spaces, Nonlinear Anal. TMA 70 (4) (2009) 1641–1647.
[13] J. Liang, G. N'Guérékata, T.J. Xiao, J. Zhang, Some properties of pseudo almost automorphic functions and applications to abstract differential equations, Nonlinear Anal. TMA 70 (7) (2009) 2731–2735.
[14] Z.H. Zhao, Y.K. Chang, G.M. N'Guérékata, Pseudo almost automorphic mild solutions to semilinear integral equations in a Banach space, Nonlinear Anal. TMA 74 (8) (2011) 2887–2894.
[15] T. Diagana, C.M. Mahop, G.M. N'Guérékata, Pseudo almost periodic solution to some semilinear differential equations, Math. Comput. Modelling 43 (2006) 89–96.
[16] M. Bahaj, O. Sidki, Almost periodic solutions of semilinear equations with analytic semigroups in Banach spaces, Electron. J. Differential Equations 2002 (98) (2002) 1–11.
[17] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, in: Applied Mathematical Sciences, vol. 44, Springer-Verlag, New York, Berlin, Heidelberg, Tokyo, 1983.
[18] Jin Liang, Jun Zhang, Ti-Jun. Xiao, Composition of pseudo almost automorphic and asymptotically almost automorphic functions, J. Math. Anal. Appl. 340 (2) (2008) 1493–1499.